Modified nonlocal elasticity theory for functionally graded materials

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ABSTRACT

In this paper, it will be shown that the nonlocal theory of Eringen is not generally suitable for analysis of functionally graded (FG) materials at micro/nano scale and should be modified. In the current work, an imaginary nonlocal strain tensor is introduced and used to directly obtain the nonlocal stress tensor. Similar to the stress tensor in Eringen’s nonlocal theory, the imaginary nonlocal strain tensor at a point is assumed to be a function of local strain tensor at all neighbor points. To compare the new modified nonlocal theory with Eringen’s theory, free vibration of FG rectangular micro/nanoplates with simply supported boundary conditions are investigated based on the first-order plate theory and three-dimensional (3-D) elasticity theory. The material properties are assumed to be functionally graded only along the plate thickness. The effects of nonlocal parameter and material gradient index on the natural frequencies of FG micro/nano plates are discussed. The present developed nonlocal theory can be used in conjunction with different analytical and numerical methods to analyze mechanical response of micro/nano structures made of FG materials.

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1. Introduction

Functionally graded materials are an advanced class of composite materials where the volume fractions of the constitutive materials are varied continuously as a function of position from one point to the other. This continuity provides continuous distribution of material properties and results in eliminating common interface difficulties of laminate composite materials (Koizumi, 1993). The advanced physical and mechanical characteristics of FG materials make them to be research matter for diverse disciplines as tribology, geology, optoelectronics, biomechanics, fracture mechanics and micro- and nanotechnology (Suresh & Mortensen, 1998; Suresh, 2001). Introducing of FG materials to micro- and nanotechnology has led to easily achieving the micro/nano devices, with better physical properties, such as micro/nano electromechanical systems (Witvrouw & Mehta, 2005; Lee et al., 2006), shape memory alloy thin films (Fu, Du, & Zhang, 2003) and atomic force microscopes (Rahaeifard, Kahrobaian, & Ahmadian, 2009).

The size dependence of mechanical behavior in micro/nano scale makes the applicability of classic continuum theory somewhat questionable. The classic continuum theory which is scale independent does not capture small size effect, and so, cannot correctly predict mechanical behavior of micro/nano structures. Hence, non-classical continuum theories such as classical couple stress (Toupin, 1962; Mindlin & Tiersten, 1962; Mindlin, 1963; Koiter, 1964), strain gradient (Aifantis, 1995; 2011).

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nonlocal elasticity \cite{Eringen1972, Eringen1972} and modified couple stress \cite{Yang2002} have been extended to accommodate the size effect of micro/nanomaterials. In the classical couple stress theories \cite{Toupin1962, Mindlin1962, Mindlin1963, Koiter1964}, the material particle can be under applied loads including not only a force to translate the material particle but also a couple to rotate it. Based on this concept, strain energy is a function of both strain and curvature tensors. Recently, the modified couple stress theory has been developed by Yang et al. \cite{Yang2002}. The two main differences between the modified couple stress theory and the classical couple stress theory are the involvement of only one material length scale parameter and that the strain energy is a function of the strain and only the symmetric part of the curvature tensor. In the strain gradient theory \cite{Aifantis1999}, it is assumed that the strain energy function depends on the gradient of the strain tensor in addition to its strain tensor. The strain gradient theory includes three length scale parameters for the dilatation gradient vector, the curvature tensor and the deviatoric stretch gradient tensor. In the nonlocal theory of Eringen \cite{Eringen1972, Eringen1972}, the stress at a point in a continuum body is assumed to be a function of the strain at all neighbor points of the domain, because of atomic forces and micro/nano scale effect.

The nonlocal theory of Eringen has been developed for isotropic and homogeneous materials. But, it has been used in many works to study different mechanical behavior of micro/nano structures made of FG materials such as micro/nano beams and plates, see for example \cite{Natarajan2012, Eltaher2012, Simsek2012, Ke2012, Wang2014, Akgoz2014, Salehipour2015}. In this paper, it will be illustrated that the theory of Eringen cannot generally be applied to analyze mechanical behavior of FG micro/nano structures and must be modified for such analysis. For this purpose, a nonlocal strain tensor is defined which is similar to the stress tensor in Eringen’s theory. The nonlocal strain tensor is directly used to obtain the nonlocal stress tensor. Based on the presented modified theory, free vibration of FG rectangular micro/nanoplates is analytically carried out using the first-order plate theory and three-dimensional elasticity. The plate is assumed to be simply supported at all edges and the material properties are functionally graded only along the plate thickness. Numerical results are presented to compare the results of the modified nonlocal theory with those of Eringen’s nonlocal theory. Contrary to that observed in the Eringen’s theory, the modified nonlocal theory predicts that by increasing nonlocal parameter and material gradient index, the results of first-order plate theory and 3-D elasticity theory are very close.

\section*{2. Nonlocal elasticity of Eringen}

According to Eringen’s nonlocal elasticity, the stress tensor at a point in a continuum body not only depends on the strain tensor at that point but also on the strains at all neighbor points of the continuum body. The constitutive equations for an isotropic homogeneous nonlocal continuum body are expressed as

\begin{equation}
\begin{aligned}
\sigma_{ij} &= \int_V \alpha(|x' - x|) t_{ij}(x') dV(x'), \quad \forall x \in V \\
t_{ij} &= C_{ijkl} e_{kl}
\end{aligned}
\end{equation}

where \(\sigma_{ij}\) and \(t_{ij}\) are the nonlocal and local stress tensors, respectively; \(e_{ij}\) is the strain tensor and \(C_{ijkl}\) is the fourth order elasticity tensor. The nonlocal kernel function \(\alpha(|x' - x|)\) incorporates into the constitutive equations the nonlocal effects of local stress at the source point \(x'\). Function \(\alpha(|x' - x|)\) depends on \(\tau = e_{ij} a / l\), in which \(e_{ij}\) is material constant, and \(a\) and \(l\) are internal and external characteristic lengths. Following experimental observations, Eringen \cite{Eringen1972} proposed the following forms of function \(\alpha(|x' - x|)\) for 2-D and 3-D problems, respectively:

\begin{align}
\alpha(|x' - x|) &= \left(\frac{2 \pi \tau^2}{\tau^2}\right)^{-1} K_0(|x' - x|/\tau) \\
\alpha(|x' - x|) &= \left(\frac{\pi \tau^2}{2} \right)^{-3/2} \exp \left(-|x' - x|^2 / \tau^2\right)
\end{align}

in which \(K_0\) is the modified Bessel function. The integral form of Eq. (1) cannot be solved easily. Hence, a differential form of the constitutive equations for homogenous materials is proposed by Eringen \cite{Eringen1972} as

\begin{equation}
\left(1 - \mu \nabla^2\right) \sigma_{ij} = t_{ij}
\end{equation}

where \(\mu = (e_{ij} a^2 / l^2)\) is nonlocal parameter and \(\nabla^2\) is Laplacian operator. Based on Eq. (3), stress at a point of continuum body depends on both strain and second gradient of strain at that point. The convenient differential form of the nonlocal Eq. (3) is broadly used in different 1, 2 and 3 dimensional mechanical problems.
3. Modified nonlocal elasticity for FG materials

Based on the concept of nonlocal elasticity, stress at a point of continuum body depends on strain at all neighbor points. But, for FG materials, Eringen’s nonlocal constitutive equations incorporate material properties, elasticity modulus and Poisson’s ratio, of neighbor points in the definition of nonlocal stress. Hence, the nonlocal theory should be modified by removing the effects of material properties at neighbor points from nonlocal constitutive equations. In view of this, the following imaginary nonlocal strain tensor is introduced:

$$\tilde{\varepsilon}_{ij} = \int_{V} \alpha(x' - x) \varepsilon_{ij}(x') \, dV(x'), \quad \forall x \in V$$

(4)

and nonlocal stress $\sigma_{ij}$ is defined directly by

$$\sigma_{ij} = C_{ijkl} \tilde{\varepsilon}_{kl}$$

(5)

in which $C_{ijkl}$ is the classic fourth order elasticity tensor. The differential form of the constitutive Eq. (4) is expressed as

$$\left(1 - \mu \nabla^2\right) \tilde{\varepsilon}_{ij} = \varepsilon_{ij}$$

(6)

Nonlocal parameter $\mu = (e_0 a)^2$ depends on the type of material. Thus, for FG materials, $\mu$ varies from one point to the other. For the homogeneous materials, the constitutive equations of the new modified nonlocal theory, Eqs. (4)-(6), are converted to the constitutive equations of Eringen’s theory, Eqs. (1)-(3). It is obvious from Eqs. (5) and (6) that the nonlocal stress at a point is independent of the gradient of material properties at that point.

4. Free vibration analysis of FG micro/nanoplate based on modified nonlocal theory

Consider a rectangular FG micro/nanoplate with a length $a$, width $b$ and thickness $h$ located along the $x$, $y$ and $z$ coordinates, respectively. The material properties including elasticity modulus $E$ and mass density $\rho$ are assumed to vary along the thickness direction based on the exponential law as

$$E(z) = E_0 \exp(\phi z)$$

$$\rho(z) = \rho_0 \exp(\phi z)$$

(7a)

(7b)

where $\phi$ is the material gradient index, and $E_0$ and $\rho_0$ are the material properties at the bottom surface of plate. The nonlocal parameter $\mu$ and Poisson’s ratio should be functions of coordinate $z$, but for simplicity and to obtain analytical solution, they are assumed to be constant. Although the exponential variation of FG materials is unlikely satisfied in practice, this distribution rule is applied to obtain analytical solution for three-dimensional elasto-dynamic equations.

Based on the nonlocal balance law, the nonlocal linear elasticity equations of motion are expressed in the absence of body forces as

$$\sigma_{ij} = \rho(z) \dot{u}_i, (i = x, y, z)$$

(8)

where $(u_x, u_y, u_z) = (u, v, w)$ are components of the displacement vector in the Cartesian coordinate system.

4.1. First-order plate theory

The first-order plate theory is based on the following displacement field:

$$u(x, y, z, t) = u_0(x, y, t) + z \psi_x(x, y, t)$$

$$v(x, y, z, t) = v_0(x, y, t) + z \psi_y(x, y, t)$$

$$w(x, y, z, t) = w_0(x, y, t)$$

(9a)

(9b)

(9c)

where $(u_0, v_0, w_0)$ are the displacements of a material point on the midplane of plate in the $(x, y, z)$ coordinate directions. The local stress components associated with the displacement field in Eq. (9) are defined as

$$\begin{bmatrix}
  t_{xx} \\
  t_{xy} \\
  t_{xz} \\
  n_{yx} \\
  n_{yz} \\
  t_{yz}
\end{bmatrix} = [C] \begin{bmatrix}
  \varepsilon_{xx} \\
  \varepsilon_{xy} \\
  \varepsilon_{xz} \\
  \varepsilon_{yx} \\
  \varepsilon_{yz}
\end{bmatrix} = [C] \begin{bmatrix}
  \frac{1}{2} (u_{0x} + u_{0y} + z \psi_{x,y} + z \psi_{y,x}) \\
  \frac{1}{2} (w_{0x} + \psi_y) \\
  \frac{1}{2} (w_{0y} + \psi_x)
\end{bmatrix}$$

(10)
where

\[
[C] = \frac{E(z)}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 & 0 \\
\nu & 1 & 0 & 0 \\
0 & 0 & (1 - \nu) & 0 \\
0 & 0 & 0 & (1 - \nu)
\end{bmatrix}
\]  
(11a)

\[
\varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \quad (i,j = x,y,z)
\]  
(11b)

The global equations of plate can be extracted by integrating elasto-dynamic Eqs. (8) through the plate thickness as

\[
N_{xx} + N_{yy} = I_0 \ddot{u} + I_1 \ddot{\psi}_x
\]  
(12a)

\[
N_{xy} + N_{yx} = I_0 \ddot{v} + I_1 \ddot{\psi}_y
\]  
(12b)

\[
Q_{xx} + Q_{yy} = I_0 \ddot{w}_0
\]  
(12c)

where

\[
I_i = \int_0^h \rho(z)z^2 dz, \quad (i = 0,1,2)
\]  
(13a)

\[
N_{ij} = \int_0^h \sigma_{ij} dz, \quad (i,j = x,y)
\]  
(13b)

\[
(Q_x, Q_y) = k_s \int_0^h (\sigma_{xz}, \sigma_{yz}) dz
\]  
(13c)

Further, multiplying Eq. (8) by \(z\) then integrating through the plate thickness and employing integration-by-part, we have

\[
M_{xx} + M_{yy} - Q_x = I_1 \ddot{u} + I_2 \ddot{\psi}_x
\]  
(14a)

\[
M_{xy} + M_{yx} - Q_y = I_1 \ddot{v} + I_2 \ddot{\psi}_y
\]  
(14b)

where

\[
M_{ij} = \int_0^h \sigma_{ij} zdz, \quad (i,j = x,y)
\]  
(15)

Eqs. (12) and (14) are equations of motion in terms of nonlocal stress components. The transverse shear correction factor \(k_s\) is introduced in the first-order theory to provide non-uniform shear rigidities in the transverse sections and is often taken \(5/6\) for the isotropic materials. For 2-D problems, Laplacian operator \(\nabla^2\) reduces to 2-D one, \(\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2\). Since the material properties of plate are only functions of thickness coordinate \(z\), thus for the present 2-D problem, the modified non-local theory is converted to Eringen’s nonlocal theory.

In order to obtain equations of motion in terms of displacement components, linear operator \((1 - \mu \nabla^2)\) is used in Eqs. (12) and (14) as

\[
N^L_{xx} + N^L_{yy} = (1 - \mu \nabla^2) \left[ I_0 \ddot{u}_0 + I_1 \ddot{\psi}_x \right]
\]  
(16a)

\[
N^L_{xy} + N^L_{yx} = (1 - \mu \nabla^2) \left[ I_0 \ddot{v}_0 + I_1 \ddot{\psi}_y \right]
\]  
(16b)

\[
Q^L_{xx} + Q^L_{yy} = (1 - \mu \nabla^2) \left[ I_0 \ddot{w}_0 \right]
\]  
(16c)

\[
M^L_{xx} + M^L_{yy} - Q^L_x = (1 - \mu \nabla^2) \left[ I_1 \ddot{u}_0 + I_2 \ddot{\psi}_x \right]
\]  
(16d)

\[
M^L_{xy} + M^L_{yx} - Q^L_y = (1 - \mu \nabla^2) \left[ I_1 \ddot{v}_0 + I_2 \ddot{\psi}_y \right]
\]  
(16e)

Here \(N^L_{ij}, Q^L_{ij}\) and \(M^L_{ij}\) are local force and moment components defined as

\[
N^L_{ij} = \int_0^h t_{ij} dz, \quad (i,j = x,y)
\]  
(17a)

\[
\left( Q^L_x, Q^L_y \right) = k_s \int_0^h (t_{xz} + t_{yz}) dz
\]  
(17b)

\[
M^L_{ij} = \int_0^h t_{ij} dz, \quad (i,j = x,y)
\]  
(17c)
From Eqs. (10), (16) and (17), the nonlocal equations of motion can be obtained in terms of the displacement components as

\[ K_1(2u_{0,xx} + (1 - v)u_{0,yy} + (1 + v)v_{0,xy}) + K_2(2\psi_{x,xx} + (1 - v)\psi_{x,yy} + (1 + v)\psi_{x,xy}) = \left(1 - \frac{\mu}{\mu_N^2}\right)[l_0u_0 + l_1\ddot{u}_x] \quad (18a) \]

\[ K_1(2v_{0,yy} + (1 - v)v_{0,xx} + (1 + v)v_{0,xy}) + K_2(2\psi_{y,yy} + (1 - v)\psi_{y,xx} + (1 + v)\psi_{y,xy}) = \left(1 - \frac{\mu}{\mu_N^2}\right)[l_0v_0 + l_1\ddot{v}_y] \quad (18b) \]

\[ k_iK_i(1 - v)\left(\nabla^2w_0 + \psi_{x,xx} + \psi_{y,yy}\right) = \left(1 - \frac{\mu}{\mu_N^2}\right)[l_0w_0] \quad (18c) \]

\[ K_2(2u_{0,xx} + (1 - v)u_{0,yy} + (1 + v)v_{0,xy}) + K_3(2\psi_{x,xx} + (1 - v)\psi_{x,yy} + (1 + v)\psi_{x,xy}) - k_3K_3(1 - v)(w_{0,x} + \psi_x) \]

\[ = \left(1 - \frac{\mu}{\mu_N^2}\right)[l_1\ddot{u}_0 + l_2\ddot{\psi}_x] \quad (18d) \]

\[ K_2(2v_{0,yy} + (1 - v)v_{0,xx} + (1 + v)v_{0,xy}) + K_3(2\psi_{y,yy} + (1 - v)\psi_{y,xx} + (1 + v)\psi_{y,xy}) - k_3K_3(1 - v)(w_{0,y} + \psi_y) \]

\[ = \left(1 - \frac{\mu}{\mu_N^2}\right)[l_1\ddot{v}_0 + l_2\ddot{\psi}_y] \quad (18e) \]

where

\[ K_i = \int_0^\frac{1}{4} \frac{E(z)}{2(1 - \nu_2^2)}z^{i-1}dz, \quad (i = 1, 2, 3) \quad (19) \]

The simply-supported boundary conditions in the nonlocal field are in the form

\[ v_0 = \psi_y = w_0 = N_{xx} = M_{xx} = 0. \quad (x = 0, a) \quad (20a) \]

\[ u_0 = \psi_x = w_0 = N_{yy} = M_{yy} = 0. \quad (y = 0, b) \quad (20b) \]

According to the Navier approach, the solutions are found as

\[ u_0(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn} \cos(\alpha_n x) \sin(\beta_m y) e^{i\omega_{mn} t} \quad (21a) \]

\[ v_0(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn} \sin(\alpha_n x) \cos(\beta_m y) e^{i\omega_{mn} t} \quad (21b) \]

\[ w_0(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn} \sin(\alpha_n x) \sin(\beta_m y) e^{i\omega_{mn} t} \quad (21c) \]

\[ \psi_x(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \psi_{x,mn} \cos(\alpha_n x) \sin(\beta_m y) e^{i\omega_{mn} t} \quad (21d) \]

\[ \psi_y(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \psi_{y,mn} \sin(\alpha_n x) \cos(\beta_m y) e^{i\omega_{mn} t} \quad (21e) \]

here \( j = \sqrt{-1} \), \( \alpha_n = m\pi/a, \beta_m = n\pi/b \) and \( \omega_{mn} \) is the natural frequency. Substitution of displacements from Eq. (21) into Eqs. (18a)–(18e) and observing non-trivial solution leads to the eigenvalue problem, Eq. (22). The natural frequencies are extracted from this equation.

\[
\begin{bmatrix}
K_{11} - M_{11}\omega_{m,n}^2 & K_{12} & 0 & K_{14} - M_{14}\omega_{m,n}^2 & K_{15} \\
K_{12} & K_{22} - M_{22}\omega_{m,n}^2 & 0 & K_{24} & K_{25} - M_{25}\omega_{m,n}^2 \\
0 & 0 & K_{33} - M_{33}\omega_{m,n}^2 & K_{34} & K_{35} \\
K_{14} - M_{14}\omega_{m,n}^2 & K_{24} & K_{34} & K_{44} - M_{44}\omega_{m,n}^2 & K_{45} \\
K_{15} & K_{25} - M_{25}\omega_{m,n}^2 & K_{35} & K_{45} & K_{55} - M_{55}\omega_{m,n}^2 \\
\end{bmatrix} = 0 \quad (22)
\]
where
\[
\begin{align*}
K_{11} &= -K_1 (2x_n^m + (1 - v)\beta^2_n) \\
K_{12} &= -K_1 (1 + v)\sigma_m\beta_n \\
K_{14} &= -K_2 (2x^2_n + (1 - v)\beta^2_n) \\
K_{15} &= K_{24} = -K_2 (1 + v)\sigma_m\beta_n \\
K_{22} &= -K_1 (2\beta^2_n + (1 - v)x^2_n) \\
K_{25} &= -K_2 (2\beta^2_n + (1 - v)x^2_n) \\
K_{33} &= -k_n K_1 (1 - v)(x^2_m + \beta^2_n) \\
K_{34} &= -K_n K_1 \sigma_m (1 - v) \\
K_{35} &= -K_n K_1 \beta_n (1 - v) \\
K_{44} &= -K_3 (2x_n^m + (1 - v)\beta^2_n) - k_n K_1 (1 - v) \\
K_{45} &= -K_3 (1 + v)\sigma_m\beta_n \\
K_{55} &= -K_3 (2\beta^2_n + (1 - v)x^2_n) - k_n K_1 (1 - v) \\
M_{11} &= M_{22} = M_{33} = -I_0 (1 + \mu(x^2_m + \beta^2_n)) \\
M_{14} &= M_{25} = -I_1 (1 + \mu(x^2_m + \beta^2_n)) \\
M_{44} &= M_{55} = -I_2 (1 + \mu(x^2_m + \beta^2_n))
\end{align*}
\] (23)

4.2. Three-dimensional elasticity

Based on the constitutive equations of modified nonlocal theory, the equations of motion (8) can be expressed as
\[
\frac{[C]_{ijkl}[\varepsilon_{kl}]}{\rho(z)} = \ddot{u}_i, \quad (i = x, y, z)
\] (24)

where the tensor $[C]_{ijkl}$ in three-dimensional elasticity is given as
\[
[C] = \frac{E(z)}{1 + v} \begin{bmatrix}
\frac{1 - \nu}{1 - 2\nu} & \frac{\nu}{1 - 2\nu} & \frac{\nu}{1 - 2\nu} & 0 & 0 & 0 \\
\frac{\nu}{1 - 2\nu} & \frac{1 - \nu}{1 - 2\nu} & \frac{\nu}{1 - 2\nu} & 0 & 0 & 0 \\
\frac{\nu}{1 - 2\nu} & \frac{\nu}{1 - 2\nu} & \frac{1 - \nu}{1 - 2\nu} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] (25)

By applying operator $(1 - \mu \nabla^2)$ on Eq. (24), we get
\[
(1 - \mu \nabla^2) \left( \frac{[C]_{ijkl}[\varepsilon_{kl}]}{\rho(z)} \right) = \left( 1 - \mu \nabla^2 \right) \ddot{u}_i, \quad (i = x, y, z)
\] (26)

Using Eqs. (6), (25) and material properties from Eq. (7) in Eq. (26), yields the following equations of motion:
\[
\frac{E_0}{\rho_0 (1 + v)} \left[1 - \nu \right] \dot{u}_{xx} + \nu \dot{u}_{yy} + \nu \dot{u}_{zz} = \left(1 - \mu \nabla^2 \right) \ddot{u} \\
\frac{E_0}{\rho_0 (1 + v)} \left[1 - \nu \right] \dot{u}_{yy} + \nu \dot{u}_{xx} + \nu \dot{u}_{zz} + \phi \frac{\nu}{1 - 2\nu} \dot{u}_{zz} + \phi \frac{\nu}{1 - 2\nu} \dot{u}_{yy} = \left(1 - \mu \nabla^2 \right) \ddot{u} \\
\frac{E_0}{\rho_0 (1 + v)} \left[1 - \nu \right] \dot{u}_{zz} + \nu \dot{u}_{xx} + \nu \dot{u}_{yy} + \phi \frac{\nu}{1 - 2\nu} \dot{u}_{zz} + \phi \frac{\nu}{1 - 2\nu} \dot{u}_{xx} + \phi \frac{\nu}{1 - 2\nu} \dot{u}_{yy} = \left(1 - \mu \nabla^2 \right) \ddot{u}
\] (27a–27c)

Substituting strain components from Eq. (11b) into Eqs. (27a)–(27c), the 3-D modified nonlocal equations of motion for an isotropic FG material are obtained in terms of displacements as:
\[
\frac{E_0}{\rho_0 (1 + v)} \left[1 - \nu \right] u_{xx} + \nu u_{yy} + \nu u_{zz} + \frac{\nu}{1 - 2\nu} W_{yx} + \frac{\nu}{1 - 2\nu} W_{yy} + \frac{\nu}{1 - 2\nu} W_{zz} + \phi \frac{\nu}{1 - 2\nu} W_{yy} + \phi \frac{\nu}{1 - 2\nu} W_{zz} = \left(1 - \mu \nabla^2 \right) \ddot{u} \\
\frac{E_0}{\rho_0 (1 + v)} \left[1 - \nu \right] u_{yy} + \nu u_{xx} + \nu u_{zz} + \frac{\nu}{1 - 2\nu} W_{yx} + \frac{\nu}{1 - 2\nu} W_{yy} + \frac{\nu}{1 - 2\nu} W_{zz} + \phi \frac{\nu}{1 - 2\nu} W_{xx} + \phi \frac{\nu}{1 - 2\nu} W_{yy} = \left(1 - \mu \nabla^2 \right) \ddot{u} \\
\frac{E_0}{\rho_0 (1 + v)} \left[1 - \nu \right] u_{zz} + \nu u_{xx} + \nu u_{yy} + \frac{\nu}{1 - 2\nu} W_{yx} + \frac{\nu}{1 - 2\nu} W_{yy} + \frac{\nu}{1 - 2\nu} W_{zz} + \phi \frac{\nu}{1 - 2\nu} W_{xx} + \phi \frac{\nu}{1 - 2\nu} W_{yy} = \left(1 - \mu \nabla^2 \right) \ddot{u}
\] (28a–28c)
In order to solve the governing vibration equations, two different displacement fields are used for the in-plane and out-of-plane vibration modes.

4.2.1. In-plane vibration analysis

For the in-plane free vibration, the following expressions for the displacement components are used:

\[ u = f(z) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_m \cos(\alpha_m x) \sin(\beta_n y) e^{i\omega_{mn}t} \]  
\[ v = g(z) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_m \sin(\alpha_m x) \cos(\beta_n y) e^{i\omega_{mn}t} \]  
\[ w = 0 \]

where unknown functions \( f(z) \) and \( g(z) \) determine the through thickness variations of in-plane displacements. The displacement components in Eq. (29) satisfy simply supported boundary conditions given by the three-dimensional elasticity as

\[ \sigma_{xx} = v = w = 0 \quad (x = 0, a) \]  
\[ \sigma_{yy} = u = w = 0 \quad (y = 0, b) \]

Substituting the displacement field of Eqs. (29) into Eqs. (28), reduces the partial differential equations of motion to the following ordinary differential equations:

\[ (\mu \Omega_m^q - G_0) f''(z) + (-G_0 \phi) f'(z) + \left( (-\mu \gamma_{mn} + 1) \Omega_m^q + G_0 \alpha_m^2 \right) f(z) + \left( \frac{G_0 \beta_n^2}{1 - 2\nu} \right) g(z) = 0 \]  
\[ (\mu \Omega_m^q - G_0) g''(z) + (-G_0 \phi) g'(z) + \left( (-\mu \gamma_{mn} + 1) \Omega_m^q + G_0 \alpha_m^2 \right) g(z) + \left( \frac{G_0 \beta_n^2}{1 - 2\nu} \right) f(z) = 0 \]

\[ a_m^2 (2\nu \phi f'(z) + f'(z)) = -\beta_n^2 (2\nu \phi g'(z) + g'(z)) \]

where

\[ G_0 = \frac{E_0}{2(1 + \nu)}, \quad \gamma_{mn} = \alpha_m^2 + \beta_n^2, \quad \Omega_m^q = \rho_0 \left( \omega_{mn}^q \right)^2 \]

In general, the solutions of Eqs. (31) are in the following form

\[ f(z) = C \exp(\lambda z), \quad g(z) = D \exp(\lambda z) \]

From Eqs. (31c) and (33), it is concluded that

\[ D = -\frac{a_m^2 C}{\beta_n^2} \]

Using Eqs. (33) and (34) in either of Eqs. (31a) or (31b), the characteristic equation is derived as

\[ (G_0 - \mu \Omega_m^q) \lambda^2 + (G_0 \phi) \lambda + \left( \Omega_m^q + G_0 \Omega_m^q - G_0 \gamma_{mn} \right) = 0 \]

The roots of Eq. (35) are obtained as

\[ \{ \lambda_1, \lambda_2 \} = \frac{1}{2(G_0 - \mu \Omega_m^q)} \left\{ \begin{array}{c} -G_0 \phi - \sqrt{\Phi} \\ -G_0 \phi + \sqrt{\Phi} \end{array} \right\} \]

where

\[ \Phi = G_0^2 \phi^2 - 4(G_0 - \mu \Omega_m^q) (\Omega_m^q + (\mu \Omega_m^q - G_0) \gamma_{mn}) \]

Thus for \( \Phi > 0 \):

\[ f(z) = C_1 \exp(\lambda_1 z) + C_2 \exp(\lambda_2 z) \]  
\[ g(z) = -\frac{a_m^2}{\beta_n^2} (C_1 \exp(\lambda_1 z) + C_2 \exp(\lambda_2 z)) \]

and for \( \Phi < 0 \):

\[ g(z) = \exp(\eta z)(C_1 \cos(\zeta z) + C_2 \sin(\zeta z)) \]  
\[ f(z) = -\frac{a_m^2}{\beta_n^2} (\exp(\eta z)(C_1 \cos(\zeta z) + C_2 \sin(\zeta z))) \]
in which
\[
\eta = \frac{-G_0 \Phi}{2(G_0 - \mu \nu_{mn})}, \quad \zeta = \frac{\sqrt{-\Phi}}{2(G_0 - \mu \nu_{mn})}.
\] (40)

The local strain components \(e_{ij}\) are obtained by substituting the displacement field, Eqs. (29), and also Eqs. (38) and (39) into Eq. (11b). By using the local strain \(e_{ij}\) in Eq. (6), the following expressions for the strain components \(e_{ij}\) can be derived: for \(\Phi > 0:\)

\[
e_{xx} = -e_{yy} = \left( -C_1 \exp(\lambda_1 z) + \frac{C_2 \exp(\lambda_2 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_n^2 - \alpha_m^2 \sin(\nu_{mn} x) \sin(\beta_n y) e^{i\theta_{mn} x t}.
\] (41a)

\[
e_{xy} = \left( C_1 \exp(\lambda_1 z) + \frac{C_2 \exp(\lambda_2 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\beta_n^2 - \alpha_m^2}{2\beta_n} \sin(\nu_{mn} x) \cos(\beta_n y) e^{i\theta_{mn} x t}.
\] (41b)

\[
e_{xz} = \frac{1}{2} \left( C_1 \exp(\lambda_1 z) + \frac{C_2 \exp(\lambda_2 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\beta_n^2}{\beta_n} \sin(\nu_{mn} x) \cos(\beta_n y) e^{i\theta_{mn} x t}.
\] (41c)

\[
e_{yz} = -\frac{1}{2} \left( C_1 \exp(\lambda_1 z) + \frac{C_2 \exp(\lambda_2 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\beta_n^2}{\beta_n} \sin(\nu_{mn} x) \cos(\beta_n y) e^{i\theta_{mn} x t}.
\] (41d)

\[
e_{zz} = 0
\] (41e)

and for \(\Phi < 0:\)

\[
e_{xx} = -e_{yy} = ( -R_1 \exp(\eta z) \cos(\zeta z) - R_2 \exp(\eta z) \sin(\zeta z)) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_n^2 - \alpha_m^2 \sin(\nu_{mn} x) \sin(\beta_n y) e^{i\theta_{mn} x t}.
\] (42a)

\[
e_{xy} = (R_1 \exp(\eta z) \cos(\zeta z) + R_2 \exp(\eta z) \sin(\zeta z)) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\beta_n^2 - \alpha_m^2}{2\beta_n} \sin(\nu_{mn} x) \cos(\beta_n y) e^{i\theta_{mn} x t}.
\] (42b)

\[
e_{xz} = \frac{1}{2} \left( (\eta R_1 + \zeta R_2) \exp(\eta z) \cos(\zeta z) + (\eta R_2 - \zeta R_1) \exp(\eta z) \sin(\zeta z) \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\beta_n^2}{\beta_n} \sin(\nu_{mn} x) \cos(\beta_n y) e^{i\theta_{mn} x t}.
\] (42c)

\[
e_{yz} = -\frac{1}{2} \left( (\eta R_1 + \zeta R_2) \exp(\eta z) \cos(\zeta z) + (\eta R_2 - \zeta R_1) \exp(\eta z) \sin(\zeta z) \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\beta_n^2}{\beta_n} \sin(\nu_{mn} x) \cos(\beta_n y) e^{i\theta_{mn} x t}.
\] (42d)

\[
e_{zz} = 0
\] (42e)

where
\[
R_1 = \frac{(1 + \mu \nu_{mn} - \mu(\eta^2 - \zeta^2))C_1 + (2\mu_\zeta)C_2}{(1 + \mu \nu_{mn} - \mu(\eta^2 - \zeta^2))^2 + (2\mu_\zeta)^2}, \quad R_2 = \frac{(1 + \mu \nu_{mn} - \mu(\eta^2 - \zeta^2))C_2 - (2\mu_\zeta)C_1}{(1 + \mu \nu_{mn} - \mu(\eta^2 - \zeta^2))^2 + (2\mu_\zeta)^2}.
\] (43)

The bottom and top surfaces of plate are free of traction, therefore:
\[
\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0 \left( z = -h \frac{h}{2} \right)
\] (44)

By substituting Eqs. (41) and (42) into Eq. (5) and using Eq. (25), the lateral boundary conditions, Eq. (44), are converted to two independent algebraic equations for the cases \(\Phi > 0\) and \(\Phi < 0\) as follows:

for \(\Phi > 0:\)

\[
E(z) \left( \frac{C_1 \exp(\lambda_1 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} + \frac{C_2 \exp(\lambda_2 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} \right) \bigg|_{z = -h/2} = 0
\] (45a)

\[
E(z) \left( \frac{C_1 \exp(\lambda_1 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} + \frac{C_2 \exp(\lambda_2 z)}{1 + \mu(\nu_{mn} - \lambda_1^2)} \right) \bigg|_{z = h/2} = 0
\] (45b)
and for $\Phi < 0$:
\[
E(z)(\eta R_1 + \zeta R_2)\exp(\eta z)\cos(\zeta z) + (\eta R_2 - \zeta R_1)\exp(\eta z)\sin(\zeta z))|_{z=\frac{1}{2}} = 0
\]
\[
E(z)(\eta R_1 + \zeta R_2)\exp(\eta z)\cos(\zeta z) + (\eta R_2 - \zeta R_1)\exp(\eta z)\sin(\zeta z))|_{z=\frac{3}{2}} = 0
\]
(46)

To obtain nontrivial solution, $C_i \neq 0$, and extract the natural frequencies of in-plane modes, the determinant of coefficient matrices are set to zero. The roots of characteristic equations yield the natural frequencies of in-plane modes.

4.2.2. Out-of-plane vibration analysis

In order to solve equations of motion for the out-of-plane free vibration modes, the following displacement field is used:
\[
\begin{align*}
  u &= f(z) \sum_{n=1}^{\infty} \alpha_n \cos(\alpha_n x) \sin(\beta_n y) e^{i \omega_n z t} \\
  v &= f(z) \sum_{n=1}^{\infty} \beta_n \sin(\alpha_n x) \cos(\beta_n y) e^{i \omega_n z t} \\
  w &= g(z) \sum_{n=1}^{\infty} \sin(\alpha_n x) \sin(\beta_n y) e^{i \omega_n z t}
\end{align*}
\]
(47a) (47b) (47c)

Employing Eqs. (47a)–(47c) in Eqs. (28a)–(28c) and simplifying the consequent results give only two independent ordinary differential equations in the following form
\[
\begin{align*}
  A_1 f''(z) + A_2 f'(z) + A_3 f(z) + A_4 g(z) &= 0 \\
  B_1 g''(z) + B_2 g'(z) + B_3 g(z) + B_4 f'(z) + B_5 f(z) &= 0
\end{align*}
\]
(48a) (48b)
in which
\[
\begin{align*}
  A_1 &= G_0 - \mu \Omega_{mn}^2, & A_2 &= A_5 = G_0 \phi, & A_3 &= \Omega_{mn}^2 + \left(\mu \Omega_{mn}^2 - \frac{2G_0(1-v)}{1-2v}\right) \gamma_{mn}, & A_4 &= \frac{G_0}{1-2v} \\
  B_1 &= \frac{2G_0(1-v)}{1-2v} - \mu \Omega_{mn}^2, & B_2 &= \frac{G_0(1-v)}{1-2v}, & B_3 &= \Omega_{mn}^2 + \left(\mu \Omega_{mn}^2 - G_0\right) \gamma_{mn} \\
  B_4 &= -\frac{G_0}{1-2v} \gamma_{mn}, & B_5 &= -\frac{2G_0 \phi \gamma_{mn}}{1-2v}
\end{align*}
\]
(49)

The general solutions of Eqs. (48a) and (48b) have the following exponential form
\[
\begin{align*}
  f(z) &= K \exp(\lambda z), & g(z) &= L \exp(\lambda z)
\end{align*}
\]
(50)

Substituting Eq. (50) into Eqs. (48a) and (48b) and simplifying results in a matrix equation as
\[
\begin{bmatrix}
  A_1 \lambda^2 + A_2 \lambda + A_5 \\
  B_4 \lambda + B_5
\end{bmatrix}
\begin{bmatrix}
  K \\
  L
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]
(51)

For nontrivial solution, determinant of the matrix should be set to zero, thus:
\[
\begin{bmatrix}
  A_1 \lambda^2 + A_2 \lambda + A_5 \\
  B_4 \lambda + B_5
\end{bmatrix}
= 0
\]
(52)
The roots of the resulted characteristic equation are obtained as
\[
\begin{align*}
  \lambda_1 &= \frac{-M_2}{4M_1} + \frac{1}{2} \sqrt{\Phi_1} + \frac{1}{2} \sqrt{\Phi_2} \\
  \lambda_2 &= \frac{-M_2}{4M_1} - \frac{1}{2} \sqrt{\Phi_1} - \frac{1}{2} \sqrt{\Phi_2} \\
  \lambda_3 &= \frac{-M_2}{4M_1} - \frac{1}{2} \sqrt{\Phi_1} + \frac{1}{2} \sqrt{\Phi_2} \\
  \lambda_4 &= \frac{-M_2}{4M_1} + \frac{1}{2} \sqrt{\Phi_1} - \frac{1}{2} \sqrt{\Phi_2}
\end{align*}
\]
(53)
where
\[
\begin{align*}
  \Phi_1 &= \frac{M_2^2}{4M_1^2} - \frac{2M_3}{3M_1} + \frac{(\Phi_3 + \sqrt{\Phi_4})^{1/3}}{32^{1/3}M_1} + \frac{2^{1/3} \left(M_2^2 - 3M_2 M_4 + 12M_1 M_5\right)}{3M_1 (\Phi_3 + \sqrt{\Phi_4})^{1/3}} \\
  \Phi_2 &= \frac{3M_3^2}{4M_1^2} - \Phi_1 + \left(\frac{-M_2^2}{M_1^2} + \frac{4M_2 M_3}{M_1^2} - \frac{8M_4}{M_1}\right)^{1/3}/4 \sqrt{\Phi_1}
\end{align*}
\]
(54)
and

\[
\begin{align*}
M_1 &= A_1 B_1 \\
M_2 &= A_1 B_2 + A_2 B_1 \\
M_3 &= A_1 B_3 + A_2 B_2 + A_3 B_1 - A_4 B_4 \\
M_4 &= A_2 B_3 + A_3 B_2 - A_4 B_5 - A_5 B_4 \\
M_5 &= A_2 B_5 - A_5 B_5 \\
\Phi_1 &= 2M_3^2 - 9M_2 M_4 - 72M_1 M_3 M_5 + 27\left(M_1 M_4^2 + M_5 M_2^2\right) \\
\Phi_2 &= \Phi_3 - 4\left(M_2^2 - 3M_2 M_4 + 12M_1 M_5\right)^3
\end{align*}
\]  

(55)

The roots in Eq. (53) are real or complex. When the roots are real, we get

\[
\begin{align*}
f(z) &= K_1 \exp(\lambda_1 z) + K_2 \exp(\lambda_2 z) + K_3 \exp(\lambda_3 z) + K_4 \exp(\lambda_4 z) \\
g(z) &= L_1 \exp(\lambda_1 z) + L_2 \exp(\lambda_2 z) + L_3 \exp(\lambda_3 z) + L_4 \exp(\lambda_4 z)
\end{align*}
\]  

(56a)

By substituting Eqs. (56) into either of Eqs. (48a) or (48b), the coefficient \(L_i\) can be obtained in terms of \(K_i\) as

\[
\begin{align*}
L_1 &= -\frac{A_1 \lambda_1^2 + A_2 \lambda_1 + A_3}{A_4 \lambda_1 + A_5} K_1 \\
L_2 &= -\frac{A_1 \lambda_2^2 + A_2 \lambda_2 + A_3}{A_4 \lambda_2 + A_5} K_2 \\
L_3 &= -\frac{A_1 \lambda_3^2 + A_2 \lambda_3 + A_3}{A_4 \lambda_3 + A_5} K_3 \\
L_4 &= -\frac{A_1 \lambda_4^2 + A_2 \lambda_4 + A_3}{A_4 \lambda_4 + A_5} K_4
\end{align*}
\]  

(57)

For the roots with complex values, functions \(f(z)\) and \(g(z)\) must be changed accordingly. But, to be concise, the new forms of these functions are not presented here.

Using Eqs. (6), (11b), (47) and (56), the strain components \(\tau_{ij}\) are obtained as

\[
\begin{align*}
\begin{pmatrix}
\tau_{xx} \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{pmatrix} &= \begin{pmatrix}
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z))
\end{pmatrix} \times 
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix}
-\frac{\chi_n^2}{\beta_n^2} \sin(\chi_n z) \sin(\beta_n y) \\
-\frac{\chi_n}{\beta_n} \sin(\chi_n z) \cos(\beta_n y) \\
\chi_n \beta_n \cos(\chi_n z) \cos(\beta_n y) \\
-\frac{\chi_n}{\beta_n} \sin(\chi_n z) \cos(\beta_n y)
\end{pmatrix} e^{\nu \beta_n^2 z}
\end{align*}
\]  

(58a)

\[
\begin{align*}
\begin{pmatrix}
\frac{\partial \tau_{xz}}{\partial z} \\
\frac{\partial \tau_{yz}}{\partial z}
\end{pmatrix} &= \frac{1}{2} \begin{pmatrix}
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z))
\end{pmatrix} \times 
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \begin{pmatrix}
\chi_n \cos(\chi_n z) \sin(\beta_n y) \\
\beta_n \sin(\chi_n z) \cos(\beta_n y)
\end{pmatrix} e^{\nu \beta_n^2 z}
\end{align*}
\]  

(58b)

\[
\tau_{xz} = \begin{pmatrix}
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(\chi_n z) \sin(\beta_n y) e^{\nu \beta_n^2 z}
\end{pmatrix}
\]  

(58c)

where

\[
\chi_n(z) = \frac{\exp(\rho z)}{1 + \mu(\gamma_{mn} - \rho)}, \quad (i = 1, 2, 3, 4)
\]  

(59)

It should be noted that according to Eq. (44) the shear and normal tractions at the lateral surfaces are zero. By inserting Eqs. (58) into Eq. (5) and using Eq. (25), the six boundary conditions of Eq. (44) are changed to the following four independent algebraic equations:

\[
E(z)\left(\begin{array}{c}
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z)) \\
(K_1 \chi_1(z) + K_2 \chi_2(z) + K_3 \chi_3(z) + K_4 \chi_4(z))
\end{array}\right) = 0. \quad \left(z = -\frac{h}{2}, \frac{h}{2}\right)
\]  

(60a)

\[
E(z)\left(\begin{array}{c}
(L_1 \chi_1(z) + L_2 \chi_2(z) + L_3 \chi_3(z) + L_4 \chi_4(z)) \\
(L_1 \chi_1(z) + L_2 \chi_2(z) + L_3 \chi_3(z) + L_4 \chi_4(z)) \\
(L_1 \chi_1(z) + L_2 \chi_2(z) + L_3 \chi_3(z) + L_4 \chi_4(z)) \\
(L_1 \chi_1(z) + L_2 \chi_2(z) + L_3 \chi_3(z) + L_4 \chi_4(z))
\end{array}\right) = 0. \quad \left(z = -\frac{h}{2}, \frac{h}{2}\right)
\]  

(60b)
Eqs. (60) are four algebraic equations in terms of four coefficients $K_i$. In order to obtain the natural frequencies of the out-of-plane modes, the determinant of coefficient matrix must set to zero.

5. Numerical examples and discussion

For the sake of generality and convenience, the following non-dimensional parameters are used:

$$\bar{\phi} = \frac{E_{(h/2)}}{E_{(-h/2)}} = \exp(\phi h)$$ \hspace{1cm} (61a)

$$\bar{\mu} = \mu/h^2$$ \hspace{1cm} (61b)

$$\omega_{m,n} = \alpha n a^2 \sqrt{\rho/E}/h$$ \hspace{1cm} (61c)

where $E_{(-h/2)}$ and $E_{(h/2)}$ are the elasticity modulus at the bottom and top surfaces of plate, respectively.

To the best of author's knowledge, the results of FG micro/nanoplates based on the experimental data or atomistic methods are not available in the literature. Hence, the comparison studies are carried out between the results corresponding to theory of Eringen, 3-D elasticity solution, and the present modified nonlocal theory. Based on the modified nonlocal theory, the non-dimensional natural frequencies $\bar{\omega}_{1,1}$ of a square FG micro/nanoplate are tabulated in Table 1 and compared with the results given by Salehipour et al. (2015). The bold-face values denote the natural frequencies related to the in-plane modes. It can be observed that for $\mu = 0$, the results of present 3-D elasticity solution are the same as that presented by Salehipour et al. (2015). As expected, the natural frequency of the in-plane mode for the present 2-D (first order) and 3-D solutions are the same and does not change with the material gradient index. But, Eringen's nonlocal theory (3-D solution) predicts higher frequency values for in-plane mode rather than present solutions. As shown in the table, for all three modes,

<table>
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<tr>
<th>$\mu$</th>
<th>$\phi$</th>
<th>$\alpha/h$</th>
<th>Method</th>
<th>$q$ (mode number)</th>
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<td></td>
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<td>1</td>
</tr>
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<td>Present (3-D elasticity)</td>
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<td></td>
<td>Salehipour et al. (2015)</td>
<td>5.7108</td>
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<td>Present (First-order)</td>
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<td>Present (3-D elasticity)</td>
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the differences between the results of present study and those of Eringen’s theory (3-D solution) increase when the nonlocal parameter or gradient index increase.

Figs. 1 and 2 are drawn to display variation of non-dimensional fundamental frequency of square FG micro/nanoplates versus the material gradient index based on the present 2-D (first-order) nonlocal theory, 3-D modified nonlocal theory (elasticity solution) and also Eringen’s theory (3-D elasticity solution). The curves are plotted for the nonlocal parameter values \( \mu = 0.1 \) and 0.2, and the length-to-thickness ratios of 5 and 10. It is clearly seen that based on the Eringen’s theory (3-D elasticity solution), the fundamental frequency rises when the gradient index increases. But, the other theories predict completely different behavior. In other words, the 2-D nonlocal theory (first-order) and 3-D modified nonlocal theory illustrate that the frequency slowly decreases when the gradient index increases. The unexpected results of the Eringen’s theory (3-D elasticity solution) which have large differences with the results of other theories, originate from the fact that the operator \( \nabla^2 \) in Eringen’s nonlocal theory is applied on both material properties and strain tensor.

Figs. 3 and 4 are plotted to show the variation of fundamental frequency of FG micro/nanoplates versus nonlocal parameter, using aforementioned theories for the length-to-thickness ratios of 5 and 10, respectively. The material gradient index values are set to be 2 and 3. It is evident that by increasing the nonlocal parameter, the differences between the results...
obtained by different aforementioned theories increase. But, the differences between the results of Eringen’s theory (3-D elasticity solution) and that of other two theories are more significant. It is seen from the figures that for higher value of the gradient index or for thinner plates, the variation of frequency with nonlocal parameter based on the Eringen’s theory (3-D elasticity solution) is completely different from that predicted by the other two theories. It is concluded from Figs. 1–4 that for sufficiently large values of gradient index and nonlocal parameter, the fundamental frequency corresponding to Eringen’s theory (3-D elasticity solution) takes very higher values. This is attributed to the important role of the material gradient index in the Eringen’s nonlocal theory.

6. Conclusion

The present study serves to propose a size-dependent modified theory based on the nonlocal Eringen’s theory for analysis of structures made of FG materials. The modified theory eliminates the couple effect of nonlocal parameter and gradient of material properties. In view of this, the nonlocal stress components are directly obtained from a new nonlocal strain tensor which is a function of the neighbor strain tensors in the continuum body. Based on the modified nonlocal theory and utilizing first-order plate theory and three-dimensional elasticity theory, analytical solutions are obtained for free vibration of FG micro/nanoplates. Numerical examples are carried out to compare the results of present study with the results obtained from
three-dimensional Eringen's nonlocal elasticity theory. It is observed that for sufficiently large values of the nonlocal parameter and gradient index, the results of three-dimensional Eringen's theory are much higher than that of present solutions. The presented modified nonlocal theory can be applied as a useful size-dependent theory in the future researches to study mechanical behavior of micro/nano structures made of FG materials.

References